Problem 1 : Flux Integral
Compute the following line integral:

1. $\vec{v} = \begin{pmatrix} x+y \\ 2y \end{pmatrix}$, $C : \vec{\gamma}(t) = (t, t^2), 0 \le t \le 1$, $\vec{N}$ the upward normal
2. $\vec{v} = \begin{pmatrix} xy^2 \\ x^2y \end{pmatrix}$, C: unit circle, $\vec{N}$ the outward normal

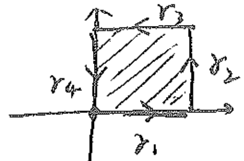
1. $\vec{r}'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix}$ $\quad \vec{n} = \begin{pmatrix} -2t \\ 1 \end{pmatrix}$

$$\int_0^1 \vec{v} \cdot \vec{n} \cdot dt = \int_0^1 \begin{pmatrix} t+t^2 \\ 2t^2 \end{pmatrix} \cdot \begin{pmatrix} -2t \\ 1 \end{pmatrix} dt = \int_0^1 -2t^3 \, dt = -\frac{1}{2}$$

2. See 5.l. very similar.

Problem 2 : Green Theorem
Compute the following line integral in two ways: by definition and by Green's Theorem:

1. Page 159 5.a
2. 5. c
3. 5. k
4. 5. l



1\. 1) $\vec{F} = \begin{pmatrix} xy \\ xy \end{pmatrix}$

$\vec{\gamma}_1(t) = \begin{pmatrix} t \\ 0 \end{pmatrix} \quad 0 \le t \le 1 \qquad \vec{\gamma}_1{}'(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \int_0^1 \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt = 0$

$\vec{\gamma}_2(t) = \begin{pmatrix} 1 \\ t \end{pmatrix} \quad 0 \le t \le 1 \qquad \vec{\gamma}_2{}'(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \int_0^1 \begin{pmatrix} t \\ t \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt = \frac{1}{2}$

$\vec{\gamma}_3(t) = \begin{pmatrix} 1-t \\ 1 \end{pmatrix} \quad 0 \le t \le 1 \qquad \vec{\gamma}_3{}'(t) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \qquad \int_0^1 \begin{pmatrix} 1-t \\ 1-t \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} dt = -\frac{1}{2}$

$\vec{\gamma}_4(t) = \begin{pmatrix} 0 \\ 1-t \end{pmatrix} \quad 0 \le t \le 1 \qquad \vec{\gamma}_4{}'(t) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \qquad \int_0^1 \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} dt = 0$

$0 + \frac{1}{2} - \frac{1}{2} + 0 = 0$

2). by Green $\quad Q_x - P_y = y - x$

$$\iint_D (y-x)\,dA = \int_0^1 \int_0^1 (y-x)\,dx\,dy = \int_0^1 \left(yx - \frac{x^2}{2}\right)\Big|_{x=0}^{x=1} dy = \int_0^1 \left(y - \frac{1}{2}\right) dy = 0$$

2\. 1) $\vec{F} = \begin{pmatrix} y\cos x \\ y\sin x \end{pmatrix}$

$\vec{\gamma}_1(t) = \begin{pmatrix} \frac{\pi}{2} t \\ 1 \end{pmatrix} \quad 0 \le t \le 1 \qquad \vec{\gamma}_1{}'(t) = \begin{pmatrix} \frac{\pi}{2} \\ 0 \end{pmatrix} \qquad \int_0^1 \begin{pmatrix} \cos\frac{\pi}{2}t \\ \sin\frac{\pi}{2}t \end{pmatrix} \cdot \begin{pmatrix} \frac{\pi}{2} \\ 0 \end{pmatrix} dt = 1$

$\vec{\gamma}_2(t) = \begin{pmatrix} \frac{\pi}{2} \\ 1+t \end{pmatrix} \quad 0 \le t \le 1 \qquad \vec{\gamma}_2{}'(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \int_0^1 \begin{pmatrix} 0 \\ 1+t \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt = \frac{3}{2}$

$\vec{\gamma}_3(t) = \begin{pmatrix} \frac{\pi}{2}(1-t) \\ 2 \end{pmatrix} \quad 0 \le t \le 1 \qquad \vec{\gamma}_3{}'(t) = \begin{pmatrix} -\frac{\pi}{2} \\ 0 \end{pmatrix} \qquad \int_0^1 \begin{pmatrix} 2\cos[(1-t)\frac{\pi}{2}] \\ 2\sin[(1-t)\frac{\pi}{2}] \end{pmatrix} \cdot \begin{pmatrix} -\frac{\pi}{2} \\ 0 \end{pmatrix} dt = -2$

$\vec{\gamma}_4(t) = \begin{pmatrix} 0 \\ 2-t \end{pmatrix} \quad 0 \le t \le 1 \qquad \vec{\gamma}_4{}'(t) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \qquad \int_0^1 \begin{pmatrix} 2-t \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} dt = 0$

$1 + \frac{3}{2} - 2 + 0 = \frac{1}{2}$

2) $Q_x - P_y = y\cos x - \cos x$

$$\int_0^{\frac{\pi}{2}} \int_1^2 (y\cos x - \cos x)\,dy\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{2}\cos x\,dx = \frac{1}{2}$$

3. 1) $\vec{r}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \quad 0 \le t \le 2\pi$

$\vec{F} = \begin{pmatrix} x^2 y \\ -xy^2 \end{pmatrix}$

$$\oint_{\partial D} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \begin{pmatrix} \cos^2 t \cdot \sin t \\ -\cos t \cdot \sin^2 t \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} dt$$

$$= \int_0^{2\pi} -2 \sin^2 t \cos^2 t \, dt$$

$$-2 \sin^2 t \cdot \cos^2 t = \frac{\sin^2 2t}{-2} = \frac{1}{-2} \cdot \frac{1 - \cos 4t}{2}$$

$$\int_0^{2\pi} -2 \sin^2 t \cos^2 t \, dt = -\frac{1}{4} \cdot \int_0^{2\pi} (1 - \cos 4t) \, dt$$

$$= -\frac{\pi}{2}$$

2). $Q_x - P_y = -y^2 - x^2$

$$\iint_D -(y^2 + x^2) \, dA = \int_0^{2\pi} \int_0^1 (-r^2) \cdot r \, dr \, d\theta$$

$$= -\frac{1}{4} \cdot 2\pi = -\frac{\pi}{2}$$

4. 1) $\vec{r}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \qquad \vec{r}'(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$

$\vec{N} = \begin{pmatrix} \cos t \\ -(-\sin t) \end{pmatrix}$ or $\begin{pmatrix} -\cos t \\ -\sin t \end{pmatrix}$

we need outward direction, so

$\vec{N} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

$\cancel{\oint_{\partial D} \vec{F} \cdot d\vec{s} = \int_0^{2\pi}}$

$$\oint_{\partial D} \vec{v} \cdot \vec{N} \, ds = \int_0^{2\pi} \begin{pmatrix} \cos t \cdot \sin^2 t \\ \cos^2 t \cdot \sin t \end{pmatrix} \cdot \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \cdot \|\vec{r}'(t)\| \, dt$$

$$= \int_0^{2\pi} 2 \cos^2 t \sin^2 t \, dt = 2\left(-\frac{\pi}{2(-2)}\right) = \frac{\pi}{2}$$

2). $P_x + Q_y = y^2 + x^2$

$$\iint_D (x^2 + y^2) \, dA = \frac{\pi}{2}$$