Problem 1 : Surface Integral Part 1: Area, Mass

Given the sphere $\vec{x}(\theta, \phi) = \begin{pmatrix} R\sin\phi\cos\theta \\ R\sin\phi\sin\theta \\ R\cos\phi \end{pmatrix}$, $0 \leq \phi \leq \pi$, $0 \leq \theta \leq 2\pi$,

1. What is the normal vector $\vec{x}_\theta \times \vec{x}_\phi$?
2. What is the area by surface integral?
3. What is the unit normal?
4. Suppose the density function $\mu(\theta, \phi) = \cos^2\theta$, compute the total mass of the this unit sphere?
5. What is the average z-coordinate for the upper sphere, i.e., $z \geq 0$?

Problem 2 : Surface Integral Part 2: Flux

Consider the same sphere as in part 1 with outward normal,

1. Given the vector field $\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, compute the flux?
2. Given the vector field $\vec{v} = \begin{pmatrix} y \\ -x \\ z \end{pmatrix}$, compute the flux?

Problem 3 : Divergence Theorem
Consider the same question in part 2 using divergence theorem

1. Problem 2.1
2. Problem 2.2
3. Given the vector field $\vec{v} = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix}$, what is the flux?

Problem 4 : Stokes Theorem
Consider the same sphere, and take γ to be the boundary of the upper sphere with counter-clockwise condition,

1. Given the vector field $\vec{F} = \begin{pmatrix} bz - cy \\ cx - az \\ ay - bx \end{pmatrix}$, use Stoke's theorem to compute $\int_\gamma \vec{F} \cdot d\vec{s}$.
2. Consider γ is the triangle with vertex to be $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, and $\vec{F} = \begin{pmatrix} x + y^2 \\ y + z^2 \\ z + x^2 \end{pmatrix}$, what is $\int_\gamma \vec{F} \cdot d\vec{s}$?