Homework 10, Math 401

due on April 22, 2020

Before you start, please read the syllabus carefully.

1. Let N and H be two normal subgroups of G. If $N \cap H = \{e\}$ and $G = N \cdot H$, then prove that $G \simeq N \times H$.

2. Prove that if $|G| = pq$ where $p \neq q$ are two different prime numbers with $p \not\equiv 1 \mod q$ and $q \not\equiv 1 \mod p$, then $G \simeq C_{pq}$ is a cyclic group of order pq.

3. Define *center* of a group to be the set $\{g \in G \mid \forall x \in G, gx = xg\}$. Denote the center of G by $Z(G)$.

    (a) Prove that there exists a group action of G on $X = G$ where $g * x = gxg^{-1}$.

    (b) Prove that $Z(G)$ is a normal subgroup, and $x \in Z(G)$ if and only if x forms a single orbit in X under the group action defined above.

    (c) Let G be a group of order p^α with prime number p. Prove that $|Z(G)| > 1$.

    (d) Let G be a group of order p^2 with prime number p. Prove that G is abelian.

4. Let F be a field with characteristic 0. Prove that a irreducible polynomial $f(x) \in F[x]$ has distinct roots.

5. If $K = \mathbb{Q}[\alpha]$ is a Galois extension over $\mathbb{Q}$, and $f(x)$ is an irreducible polynomial with $f(\alpha) = 0$. Denote $n = \deg(f)$. Prove that for each root α_i of $f(x)$ for $1 \leq i \leq n$, there exists a unique ring isomorphism σ such that $\sigma(\alpha) = \alpha_i$. (Hint: for existence, you can use the ring isomorphism to $\mathbb{Q}[x]/\langle f(x)\rangle$ as a bridge.)

6. Let $K = \mathbb{Q}[2^{1/3}, \zeta_3]$ be the splitting field of $f(x) = x^3 - 2$. Write down all elements in $\mathrm{Aut}(K/\mathbb{Q})$.

7. Let N be a normal subgroup of G and H be a normal subgroup of N. Is it true that H is normal in G? If yes, give a proof, if no, give a counterexample.

8. Let $K = \mathbb{Q}[\sqrt{2}, \sqrt{5}]$. Write down all elements in $\mathrm{Aut}(K/\mathbb{Q})$.

9. Let $K = \mathbb{Q}[\zeta_p]$ with a prime p. Write down all elements in $\mathrm{Aut}(K/\mathbb{Q})$.