Homework 4, Math 401

due on February 10, 2020

Before you start, please read the syllabus carefully.

1. Consider $\mathbb{Z}[\sqrt{-1}]$. As a set, it contains all elements in the form of $a + b\sqrt{-1}$ where a and b are in $\mathbb{Z}$. The addition and multiplication is defined as the same addition and multiplication in complex numbers. Prove that $\mathbb{Z}[\sqrt{-1}]$ is a commutative ring.
2. Prove that $\mathbb{Z}[\sqrt{-1}]$ is an integral domain.
3. Given a surjective ring homomorphism $\phi : A \to B$ between two commutative rings.
   (a) Denote J to be an ideal of B. Prove that $\phi^{-1}(J) := \{x \in A \mid \phi(x) \in J\}$ is an ideal of A.
   (b) Prove that if every ideal of A is principal, then every ideal of B is principle.
4. Find all the ideals in the ring of
   (a) $\mathbb{Z}$
   (b) $\mathbb{F}[x]$ where F is a field
   (c) $\mathbb{Z}_p$ where p is a prime
   (d) $\mathbb{Z}_{pq}$ where p and q are two different primes.
   (e) $\mathbb{Z}_{p^2}$ where p is a prime.
5. Show that $\mathbb{Z}_5$ is a quotient ring of $\mathbb{Z}_{10}$ (equivalently, this means that $\mathbb{Z}_5$ is isomorphic to a quotient ring of $\mathbb{Z}_{10}$).
6. Given R a commutative ring. Prove that $I \cdot J := \{\sum_{1 \le k \le K} i_k \cdot j_k \mid i \in I, j \in J\}$ are still ideals of R where I and J are both ideals of R.
7. For the ring of integers $\mathbb{Z}$, denote $I = \langle m \rangle$ and $J = \langle n \rangle$. You have seen in previous exercises that $I + J$ and $I \cap J$ and $I \cdot J$ are all still ideals for the same ring R. Also you have seen that all ideals of $\mathbb{Z}$ are principle. Find the generator for the following ideal:
   (a) $I + J$
   (b) $I \cap J$
   (c) $I \cdot J$

   Bonus: Which ideal is bigger between $I \cap J$ and $I \cdot J$? Can you guess when $I \cap J = I \cdot J$ for the ring $\mathbb{Z}$?
8. Find all ring homomorphisms $\phi : \mathbb{Q}[x] \to \mathbb{Q}$.

9. Prove that $\phi_a : F[x] \to F$ by mapping $\phi_a(f(x)) = f(a)$ is a surjective ring homomorphism. Determine $\text{Ker}(\phi_a)$. Show that F is a quotient ring of $F[x]$.

10. Given $I = \langle x^2 + 5 \rangle$ an ideal of $R = F[x]$. Determine R/I as a set, i.e., determine all the equivalence classes mod I.