Homework 5, Math 401

due on February 17, 2020

Before you start, please read the syllabus carefully.

1. Prove that given an integer $n = \prod_{1 \le i \le k} p_i^{r_i}$ where p_i are different primes, then
$$\mathbb{Z}_n \simeq \mathbb{Z}_{p_1^{r_1}} \times \cdots \times \mathbb{Z}_{p_k^{r_k}}.$$
2. In the above question, if $n = p_1 p_2$, determine the isomorphism map $\phi : \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \to \mathbb{Z}_{p_1 p_2}$.
3. Prove that the two definitions of $I = \langle \alpha_1, \cdots, \alpha_n \rangle$ are the same:
   1) the minimal ideal contain α_i for all i;
   2) the set $\{\sum_{1 \le i \le n} r_i \alpha_i \mid r_i \in R\}$.
4. Given an irreducible polynomial $f(x) \in F[x]$. Prove that $F[x]/\langle f(x) \rangle$ is a field.
5. Given $f(x) = x^2 + 5$, determine whether $f(x)$ is irreducible in
   (a) $\mathbb{Z}_3[x]$
   (b) $\mathbb{Z}_5[x]$
   (c) $\mathbb{Z}_7[x]$
6. Prove that $\mathbb{Z}_3[x]/\langle x^2 + 5 \rangle \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$.
7. We have learnt that $F = \mathbb{Q}[\sqrt{2}]$ is a field extension of $\mathbb{Q}$. Determine whether $x^2 + 5 \in F[x]$ is irreducible.
8. Prove by induction that it follows from Fundamental Theorem of Algebra that every $f(x) \in \mathbb{C}[x]$ can be written into a product of linear polynomials in $\mathbb{C}[x]$.