Homework 7, Math 401

due on March 16, 2020

Before you start, please read the syllabus carefully.

1. Determine the splitting field of $f(x) = x^5 - 2 \in \mathbb{Q}[x]$.
2. Consider $f(x) = x^8 - 1 \in \mathbb{Q}[x]$.
   (a) Determine the factorization of $f(x)$ into product of irreducible polynomials over $\mathbb{Q}[x]$.
   (b) Determine the splitting field K of $x^4 + 1 \in \mathbb{Q}[x]$.
   (c) Is $x^2 + 1$ irreducible in $K[x]$?
   (d) Determine the splitting field of $f(x) \in \mathbb{Q}[x]$.
3. Denote $K = \mathbb{Q}[\sqrt{2} + \sqrt{5}] \subset \mathbb{C}$.
   (a) Prove that $K \subset F := \mathbb{Q}[\sqrt{2}, \sqrt{5}]$.
   (b) Determine $[F : \mathbb{Q}]$.
   (c) Denote $\alpha = \sqrt{2} + \sqrt{5}$. Prove that $1, \alpha, \alpha^2, \alpha^3$ are linearly independent over $\mathbb{Q}$.
   (d) Prove that $K = F$.
   (e) Prove that $\{1, \alpha, \alpha^2, \alpha^3\}$ is a basis for K as a vector space over $\mathbb{Q}$.
   (f) Write α^4 as a linear combination of $\{1, \alpha, \alpha^2, \alpha^3\}$,i.e., find $a, b, c, d \in \mathbb{Q}$ such that
   $$\alpha^4 = a + b\alpha + c\alpha^2 + d\alpha^3.$$
   (g) Prove that $\mathbb{Q}[x]/\langle f(x)\rangle \simeq K$ where $f(x) = x^4 - (a + bx + cx^2 + d^3)$.
4. (a) Prove that $g(x) = x^4 + 1$ is reducible over $\mathbb{F}_3$. (Hint: find $h(x)|g(x)$)
   (b) Determine the factorization of $f(x) = x^9 - x \in \mathbb{F}_3[x]$ into irreducible polynomials.
   (c) Determine the splitting field of $f(x)$ and its degree over $\mathbb{F}_3$.