# Homework 9, Math 401

due on April 13, 2020

Before you start, please read the syllabus carefully.

1. Given $n \geq 4$.
   (a) Show that $A_n$ can be generated by all 3-cycles (meaning that the smallest subgroup containing all 3-cycles is $A_n$ itself).
   (b) Show that if $H \lhd A_n$ is a normal subgroup, and $H$ contains one 3-cycle, then $H$ contains all 3-cycle.
2. We study $G = A_4$.
   (a) Determine all subgroups of $A_4$, and determine whether they are normal or not.
   (b) Write down all increasing sequences of subgroups $G_0 = e \subset G_1 \subset G_2 \subset \cdots \subset G_n = G$ where $G_i \lhd G_{i+1}$ and $G_{i+1}/G_i$ is abelian.
3. Let $G$ be a finite group. Define a relation on $G$: $g_1 \sim g_2$ iff there exists $\sigma \in G$ such that $\sigma g_1 \sigma^{-1} = g_2$. Show that $\sim$ is an equivalence relation. (The equivalence class is called *conjugacy classes of* $G$.)
4. Let $G$ be a finite group. Define a relation on subgroups of $G$: $H_1 \sim H_2$ iff there exists $\sigma \in G$ such that $\sigma H_1 \sigma^{-1} = H_2$. Show that $\sim$ is an equivalence relation. (The equivalence class is called *conjugacy classes of subgroups of* $G$.)
5. Let $G$ be a finite group and $N \lhd G$ be a normal subgroup and $H \subset G$ be a subgroup.
   (a) We denote $N \cdot H$ to be the subset $\{n \cdot h \mid n \in N, h \in H\}$ of $G$. Show that $N \cdot H$ is a subgroup.
   (b) Show that $N \cdot H = H \cdot N$.
   (c) Show that $N \cap H$ is a normal subgroup of $H$.
   (d) Show that $N \cdot H/N \simeq H/(N \cap H)$.
6. Let $N \lhd G$ be a normal subgroup of $G$. Show that if $G$ is solvable, then $G/N$ is solvable.
7. Given a finite group $G$. Prove that if $|G| = p$ and $p$ is a prime number, then $G \simeq C_p$ where $C_p$ is the cyclic group with order $p$.
8. Given a finite group $G$. Prove that if $|G| = p^n$ some prime power, then there exists $g \in G$ with $\text{ord}(g) = p$.

9. Let $G$ be a finite group. Prove that if $G/N_1$ and $G/N_2$ are abelian, then $G/(N_1 \cap N_2)$ is also abelian. (Therefore we can define $G^{ab}$ to be the maximal quotient group $G/N$ that is abelian, we call it the *abelianization of* $G$.)

10. Denote $[G, G]$ to be the smallest subgroup of $G$ containing $g_1 g_2 g_1^{-1} g_2^{-1}$ for all $g_1, g_2 \in G$. Prove that $G/[G, G] = G^{ab}$.