Recall from last time:

$$f: S_n \longrightarrow \mathbb{Z}_2 = \{0, 1\}$$

$$\sigma \longrightarrow \#\text{ of transpositions in writing } \sigma \mod 2$$

$$\equiv \#\text{ of } \{(i,j) \mid i<j,\ \sigma(i)>\sigma(j)\} \mod 2$$

We proved that f is a grp homomorphism.

So $A_n := \ker(f)$

$(123) \in S_3$? even.

$(12)(13) \in S_3$? even

$(1234) \in S_4$? odd.

$= (14)(13)(12)$.

$(1234\cdots n) \in S_n$? $\begin{cases} \text{even if } n \text{ is odd.} \\ \text{odd if } n \text{ is even.} \end{cases}$

$= (1n)(1\ n-1)\cdots(12)$

$n-1$ transposition

Goal: Show that A_n contains no $\underbrace{\text{non-trivial}}_{\text{means } N \neq e,\ N \neq A_n}$ normal subgrp when $n \geq 5$.

Lemma: Given $\sigma, \pi \in S_n$, say $\pi = (i_1\ i_2 \cdots i_k)$

$$\sigma\pi\sigma^{-1} = (\sigma(i_1)\ \sigma(i_2) \cdots \sigma(i_k))$$

pf. $\sigma\pi\sigma^{-1}(\sigma(i_j)) = \sigma\pi(i_j) = \sigma(i_{j+1})$ or i_1 if $j=k$.

Conjugation by $\sigma \in S_n$ does not change the cycle type.

$\sigma \cdot \ \cdot \sigma^{-1}$ $\qquad$ $\pi = \pi_1 \cdots \pi_k$ disjoint cycles.

eg. $$\overset{\sigma}{(1\,2\,3)}\cdot\overset{\pi}{(6\,5\,4\,3\,2\,1)}\cdot\underset{(1\,2\,3)^{-1}}{\overset{\sigma^{-1}}{(3\,2\,1)}} = (6\,5\,4\,1\,3\,2)$$

<u>Exercise</u>. If $H \trianglelefteq A_n$ for $n \geq 4$, and H contains one 3-cycle, then $H = A_n$.

Idea: generate a lot of 3-cycles via conjugation

Thm. For $n \geq 5$, A_n contains no non-trivial normal subgrps. (Rmk. A_4 does contain non-trivial normal subgrp).

Pf: It suffices to construct one 3-cycle in H.

Assume $h \in H$ $h \neq e$. $h = \pi_1 \cdots \pi_k$

1) If h contains a long cycle, say π_1, of more than 4 elements. $\pi_1 = (1\ 2\ 3 \cdots r)$ $r \geq 4$

$\sigma = (1\,2\,3)$, then $\sigma \pi_1 \sigma^{-1} = \pi_1^{\sigma} = (1\,2\,3)(1\,2\,3 \cdots r)(3\,2\,1)$

$= (2\,3\,1\,4 \cdots r)$

$h^{\sigma} \cdot h^{-1} = \pi_1^{\sigma} \cdot \pi_1^{-1}$ because σ commute with other π_i. $i > 1$.

$= (2\,3\,1\,4 \cdots r)\cdot(r\ r{-}1 \cdots 1)$

$= (1\,2\,4)$ so $H = A_n$.

2) If h contains 2 3-cycles, $h = (1\,2\,3)(4\,5\,6)\pi_3 \cdots \pi_k$.

pick $\sigma = (3\,4\,5)$ to take conjugation.

$\sigma \cdot (1\,2\,3)(4\,5\,6) \cdot \sigma^{-1} = (1\,2\,4)(5\,3\,6)$

$h^6 \cdot h^{-1} = (124)\cdot(536)\cdot(321)(654)$

$= (16345)$ ← this is a cycle longer than or equal to 4. go back to 1).

3). if h contains only one 3-cycle, all other π_i are transpositions. h^2 is a single 3-cycle. go to exercise directly.

4). if h only contains transpositions.

$h = \pi_1 \cdots \pi_k$ $\quad$ π_i are all transpositions.

suppose $\pi_1 \cdot \pi_2 = (12)(34)$

pick

$\sigma = (124)$ $\quad$ $\sigma \cdot \pi_1 \pi_2 \sigma^{-1} = (24)(31)$

$h^\sigma \cdot h^{-1} = (\sigma \cdot \pi_1 \pi_2 \sigma^{-1}) \cdot (\pi_1 \pi_2)^{-1}$

$= (24)(31)(12)(34)$

$= (14)(23)$

Notice $\boxed{n \geq 5.}$ conjugation by $\sigma_2 = (235)$

$[(14)(23)]^{\sigma_2} = (14)(35)$

$(14)(23)\cdot(14)(35) = (235)$ go to exercise. $\square$.

Def (Solvable Group). A finite grp G is called <u>solvable</u> if $\exists$ a sequence of subgrps

$$G_0 = e \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n = G$$

s.t.

1) $G_i \trianglelefteq G_{i+1}$

2) G_{i+1}/G_i is abelian.

Coro. A_n is not solvable when $n \geq 5$.

Thm. Given $N \trianglelefteq G$, where G is finite grp. G is solvable $\Leftrightarrow$ both N and G/N are solvable.

Pf: $\Leftarrow$ If N is solvable.

$$N_0 = e \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq N_r = N$$ satisfying $N_i \trianglelefteq N_{i+1}$, N_{i+1}/N_i is abelian.

G/N is solvable.

$$Q_0 = e \subseteq Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q_s = G/N$$ satisfying $Q_i \trianglelefteq Q_{i+1}$, Q_{i+1}/Q_i is abelian.

Claim: Denote

$$\begin{array}{ccc} \rho: G & \twoheadrightarrow & G/N \\ \cup\!| & & \cup\!| \\ \rho^{-1}(Q) & & Q \\ \| & & \\ \{g \in G \mid \rho(g) \in Q\} & & \end{array}$$

$\xhookrightarrow{\text{inj}}$

$\xrightarrow[\text{surj}]{}$

then $\rho^{-1}(Q)$ is also a subgrp of G.

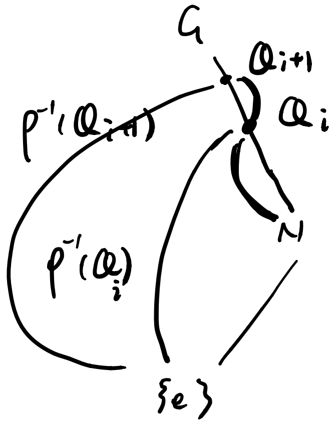
Then we claim that

$$e \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq \underset{\underset{N}{\shortparallel}}{\overset{\overset{\rho^{-1}(Q_0)}{\shortparallel}}{N_r}} \le \rho^{-1}(Q_1) \le \rho^{-1}(Q_2) \subseteq \cdots \le \rho^{-1}(Q_s) = G.$$

satisfying both conditions.

1) $N_i \triangleleft N_{i+1}$. $\quad \rho^{-1}(Q_i) \triangleleft \rho^{-1}(Q_{i+1})$.

2) $\dfrac{N_{i+1}}{N_i}$ is abelian and. $\quad \dfrac{\rho^{-1}(Q_{i+1})}{\rho^{-1}(Q_i)} \simeq \dfrac{Q_{i+1}}{Q_i}$ is abelian.



[To show the isomorphism.

$$\rho^{-1}(Q_{i+1}) \xrightarrow{\rho} \frac{Q_{i+1}}{Q_i}$$

ρ is surj grp hom with $\mathrm{Ker} = \rho^{-1}(Q_i)$

then by fundamental hom thm.

$$\frac{\rho^{-1}(Q_{i+1})}{\rho^{-1}(Q_i)} \simeq \frac{Q_{i+1}}{Q_i}$$]

[Rmk: If $e \triangleleft N \triangleleft G$ and $N \subseteq H_1 \subseteq H_2 \subseteq G$. then.

$$\frac{H_2/N}{H_1/N} \simeq \frac{H_2}{H_1}$$]

given $H_1/N \triangleleft H_2/N$. Refered to 3rd homomorphism grps.

2) "$\Rightarrow$" Suppose G is solvable

$$e \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n = G. \quad \text{s.t.}$$

1) $G_i \triangleleft G_{i+1}$

2) G_{i+1}/G_i is abelian.

then given $N \triangleleft G$,

We claim $N_i \triangleleft N_{i+1}$ and N_{i+1}/N_i is abelian.

$$e \subseteq \underset{\substack{\| \\ N_1}}{G_1 \cap N} \subseteq \underset{\substack{\| \\ N_2}}{G_2 \cap N} \subseteq \cdots \subseteq \underset{\substack{\| \\ N_n}}{G_n \cap N} = N$$

① $N_i \triangleleft N_{i+1}$ because

$\gamma \in G_{i+1}$ we know $\gamma \cdot G_i \gamma^{-1} = G_i \subseteq G_{i+1}$.

$\gamma \cdot (G_i \cap N) \gamma^{-1} \subseteq \gamma G_i \gamma^{-1} \cap \gamma N \gamma^{-1} = G_i \cap N$ then ✓.

②. N_{i+1}/N_i is abelian. because

$$f: N_{i+1} \longrightarrow G_{i+1}/G_i$$

$$n \longrightarrow nG_i$$

is a grp homomorphism with kernel $\ker(f) = N_{i+1} \cap G_i$
$= N \cap G_{i+1} \cap G_i$
$= N \cap G_i = N_i$.

So by fundamental hom thm.

$$\frac{N_{i+1}}{N_i} \simeq \operatorname{Im}(f) \subseteq G_{i+1}/G_i$$

is abelian since G_{i+1}/G_i is abelian.

[We leave the argument for G/N being solvable in the homework.]

Coro. S_n is not solvable. for $n \geq 5$.

because $A_n \triangleleft S_n$ is not solvable.