# Sylow Theorem.

Recall the theorem of Lagrange, if $H \leq G$ a subgrp of $G$. then $|H| \mid |G|$.

Q: if $n \mid |G|$, does there exist subgrp $H \leq G$ s.t $|H| = n$?

Set up $|G| = p^{\alpha} \cdot m$ $p \nmid m$.

<u>Def</u> ( Sylow $p$ subgrp) $|G| = p^{\alpha} \cdot m$ with $p \nmid m$. then a subgrp $H \leq G$ is called a Sylow-$p$ subgrp if $|H| = p^{\alpha}$.

General Answer for Q. is negative, but in this special case. $p^{\alpha} \cdot m = |G|$ $p \nmid m$. the answer is yes for $p^{\alpha}$.

If this holds. $|G| = n = p_1^{r_1} \cdot p_2^{r_2} \cdot p_3^{r_3} \cdots p_k^{r_k}$. $(p_i \neq p_j)$

suppose. for each $i$. $\exists$ $H_i \leq G$. $|H_i| = p_i^{r_i}$.

maybe $G = H_1 \times H_2 \times H_3 \times \cdots \times H_k$ ( direct product )

[ side: $|G_1 \times G_2| = |G_1| \cdot |G_2|$ ]

eg. $G = S_3$ (the smallest non-abelian finite grp.)

to see this. it suffices to go through. all finite grp with order $\leq 5$.

What is Sylow-3 subgrp of $S_3$? $\quad A_3 \ (\simeq C_3)$.

$\quad -2$ $\quad\quad G \quad\quad A_3 = \langle (123) \rangle \quad C_3 = \{0,1,2\}$.

$|S_3| = 6 = 2 \times 3$.

$G_3 = A_3 \qquad G_2 = \langle (12) \rangle$

notation for Sylow-3 subgrp of $G$. $\qquad \langle (13) \rangle$

$\qquad \langle (23) \rangle$

(Fact: there may be more than one $G_p$ for $G$ and $p$.)

① $G_3 \times G_2 \not\simeq S_3$ since there exists elements of order 6 in $C_3 \times C_2$, but no such element in $S_3$.

$G_3 \simeq C_3$, $G_2 \simeq C_2$

or ② abelian $\not\simeq$ non-abelian.

Def (nilpotent grp). $G$ is called nilpotent if

$$G \simeq \prod_{p \mid |G|} G_p .$$

Sylow Theorem: a) Sylow-$p$ subgrp exicts.

b) all Sylow-$p$ subgrps are conjugate to each other (if $H_1$, $H_2$ are both Sylow-$p$ subgrps, then $\exists\, g \in G$ s.t $gH_1g^{-1} = H_2$)

c) Denote $n_p$ to the # of Sylow-$p$ subgrps of $G$.

$n = p^{\alpha} \cdot m$. then.

$p \nmid m$

① $n_p \mid m$

② $n_p \equiv 1 \pmod p$

Tool : Grp action.

Def ( Grp action ). A grp action is grp homomorphism.

$$\phi : G \longrightarrow Perm(X) \ (\simeq S_n \text{ if } n = |X|.)$$

where the grp operation in $Perm(X)$ is composition.

Def ( transitive ) A grp action $\phi : G \to Perm(X)$ is transitive if.

$\forall\, x, y \in X, \quad \exists\, g \in G$ s.t. $\quad \phi(g)(x) = y$

in short. we will write

$$g * x = y.$$

Example. Given $H \subseteq G$. $\quad X = \{ \text{left cosets of } H \}$

$$\phi : G \longrightarrow Perm(X)$$

is this a bijection $X \to X$ ?

$$g \longrightarrow \phi(g) : \underbrace{x \cdot H}_{= \{x \cdot h \mid h \in H\},\ x \in G} \longrightarrow \underbrace{(g \cdot x) \cdot H}_{= \{g \cdot x\, h \mid h \in H\}}$$

$\phi$ is a grp action : $\qquad |G| * |X| = \dfrac{|G|}{|H|}$

① $\phi(g)$ is surjective : $\forall\, yH$ in $X$. we can find.

$g^{-1} y \cdot H$ in $X$ s.t.

$$g \cdot (g^{-1} \cdot y \cdot H) = y \cdot H$$

since $X$ has finite size. $X$ is injective.

② $\phi$ is grp homomorphism :

$$\phi(g_1 g_2) \cdot ( x \cdot H ) = g_1 g_2 \cdot x \cdot H$$

$$\phi(g_1) \circ \phi(g_2) ( x H ) = \phi(g_1) \cdot ( g_2 \cdot x H )$$

$= g_1 \cdot g_2 \cdot x \, H$

So $\phi(g_1 \cdot g_2) = \phi(g_1) \cdot \phi(g_2)$

Q: Is this $\phi$ transitive?

$\forall \, g_2 H, \, g_1 H$, we can find $g = g_1 \cdot g_2^{-1} \in G$. s.t.

$\phi(g)(g_2 H) = g_1 \cdot g_2^{-1} \cdot g_2 H = g_1 H.$

Def (stabilizer, orbit) $\phi$ is a grp action. $x \in X$.

then stabilizer $Stab_x := \{ g \in G \mid g * x = x \}$ and.

orbit $O_x := \{ y \in X \mid \exists g \text{ s.t } g * x = y \}$

Lemma: $Stab_x$ is a subgrp of $G$.

Pf: if $g_1 \in Stab_x$ then $g_1 * x = x$

$\Rightarrow g_1^{-1} * x = x.$

if $g_1 \in Stab_x$, $g_2 \in Stab_x$, then

$(g_1 \cdot g_2) * x = g_1 * (g_2 * x)$

$= g_1 * x = x$

$e \in G$. fixes every $x \in X$. D.

<u>Claim</u>: We define $x \sim y$ in $X$. if. $\exists g \in G$ s.t. $g * x = y$. Then "$\sim$" is an equivalence relation.

easy.

"~" gives a partition on $X$.

Lemma: $\phi$ is a grp action of $G$ on $X$. Then $\forall x \in X$.

$$\frac{|G|}{|Stab_x|} = |O_x|$$

Pf: There is a bijection betwen.

$\{$left cosets of $Stab_x\}$ and $\{y \mid y \in O_x\}$.

$g \cdot Stab_x$ $\qquad$ $g \cdot x$

$g \cdot s \, Stab_x$ $\qquad$ $g \cdot sx = gx = y$.

well-defined ✓

surjective ✓

injective: ✓ $\quad \forall y \in O_x$. $\quad g_1 * x = y$, $g_2 * x = y$ then $g_1^{-1} \cdot g_2 \in Stab_x$.

$$(g_1^{-1} \cdot g_2) * (x) = g_1^{-1} * (g_2 * x) = g_1^{-1} * y = x$$

Then $\dfrac{|G|}{|Stab_x|} = |O_x|$. $\square$.

Example: $X = \{$ all subgrps of $G\}$.

$\phi: G \longrightarrow Perm(X)$

$g \longrightarrow H \xrightarrow{\phi(g)} gHg^{-1}$

Check $\phi$ is a grp action: 1) $\phi(g)$ is bijective

2) $\phi(g_1 \cdot g_2) = \phi(g_1) \circ \phi(g_2)$

For 1). surj: $g^{-1}Hg \longrightarrow H$.

2) $g_1 \cdot g_2 H (g_1 \cdot g_2)^{-1} = g_1 \cdot (g_2 H g_2^{-1}) \cdot g_1^{-1}$

Def. (normalizer). Given $H \subseteq G$, we define.

$N_H := \{ g \in G \mid gHg^{-1} = H \}$.

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Fact: $N_H \supseteq H$. is another subgrp of $G$.

$\mathrm{Stab}_H = N_H$. in the conjugation action of $G$ on $\{\text{subgrps}\}$

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