Homework 1, Math 401

due on Feb 1, 2021

Before you start, please read the syllabus carefully.

1. For the following sets with the specified binary operation, do they form a group? Prove or disprove.

   (a) S : the set of continuous function over $[0,1]$ with $f(0) = f(1) = 0$
   f : function multiplication

   (b) $S : \mathbb{R}^3$
   $f : x \cdot_f y = x - y$

   (c) $S : \mathbb{Q} \backslash \{1\}$
   $f : x \cdot_f y = x + y - x \cdot y$ (where $+$ and $\cdot$ are usual addition and multiplication for rational numbers)

   (d) S : the set of orthogonal matrices in $\mathrm{GL}_2(\mathbb{R})$ (Orthogonal matrix is a matrix where $A \cdot A^T = I$)
   f: matrix multiplication

2. Given a set S, a *relation* $\sim$ is a subset of $R \subset S \times S$, i.e., $a \sim b$ if and only if $(a,b) \in R$. A relation $\sim$ is called an *equivalence relation* if:

   (a) (reflexivity) $a \sim a$ for any $a \in S$

   (b) (symmetry) $a \sim b$ if and only if $b \sim a$

   (c) (transitivity) $a \sim b$ and $b \sim c$ imply $a \sim c$.

   Prove that when S is the set of all groups, the relation $G_1 \sim G_2$ if and only if G_1 is isomorphic to G_2 is an equivalence relation.

3. Prove that the two groups $(\mathbb{Z}, +)$ and $(\mathbb{Q}, +)$ are not isomorphic to each other.

4. Let G be a group, if $a \cdot b = a \cdot c$ or $b \cdot a = c \cdot a$, then $b = c$. (This is called the cancellation law for group).

5. For the group $C_n = (\{0, 1, \cdots, n-1\}, + \mod n)$, prove that the following statement are equivalent for $k \in C_n$:

   (a) k is a generator for C_n

   (b) $\mathrm{ord}(k) = n$.

   (c) The integer k is relatively prime to n.

6. Prove that every subgroup of a finite cyclic group is still cyclic.
(Hint: if d is the greatest common divisor of a and b, then there exist integers m and n such that $am - bn = d$.)

7. Prove that if H_1 and H_2 are both subgroups of G, then $H_1 \cap H_2$ is also a subgroup of G.

   (Remark: This easy fact is assumed when we define *generation* since we need to be clear what do we mean by *the smallest subgroup* containing S).

8. (a) Let S be a nonempty subset of a group G. Prove that

$$\langle S \rangle = \{g_1^{r_1} g_2^{r_2} \cdots g_n^{r_n} \mid g_1, \cdots, g_n \in S, r_1, \cdots, r_n = \pm 1, n \geq 1\}.$$

   (b) Denote $\sigma = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$ and $\tau = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Prove that $\sigma^3 = I$, $\tau^2 = I$ and $\tau\sigma = \sigma^2\tau$, and then using the three equalities to prove that

$$\langle \sigma, \tau \rangle = \{\sigma^i \tau^j \mid 0 \leq i < 3, 0 \leq j < 2\}.$$