Homework 10, Math 401

due on April 14, 2021

Before you start, please read the syllabus carefully.

1. Let K be a finite degree extension of $\mathbb{Q}$ with $\mathrm{Aut}(K/\mathbb{Q}) = [K : \mathbb{Q}] = n$. Let $\alpha \in K$ and $\alpha_i = \sigma_i(\alpha)$ where $\mathrm{Aut}(K/\mathbb{Q}) = \{\sigma_i \mid 1 \leq i \leq n\}$. Denote $X = \{\alpha_i\}$.
   (a) Prove that $\mathrm{Aut}(K/\mathbb{Q}) \times X \to X : (\sigma, \alpha_i) = \sigma(\alpha_i)$ gives a group action on X.
   (b) Using the expression $f(x) = \prod_{1 \leq i \leq n}(x - \alpha_i)$ to determine the coefficient a_j of x^j in $f(x)$ for $0 \leq j \leq n$.
   (c) Denote $a_j(\alpha_1, \cdots, \alpha_n)$ to be a n-variable polynomial. Prove that $a_j(\alpha_1, \cdots, \alpha_n) = a_j(\alpha_{\sigma(1)}, \cdots, \alpha_{\sigma(n)})$ for any $\sigma \in S_n$.
   (d) For any $\sigma \in \mathrm{Aut}(K/\mathbb{Q})$ and $a_j(\alpha_1, \cdots, \alpha_n) \in K$ for $0 \leq j \leq n$, prove that
   $$\sigma(a_j(\alpha_1, \cdots, \alpha_n)) = a_j(\sigma(\alpha_1), \cdots, \sigma(\alpha_n)).$$
   (e) Prove that $f(x) \in \mathbb{Q}[x]$.
2. Consider $K = \mathbb{F}_{25}$.
   (a) Prove that $K \simeq \mathbb{F}_5[x]/\langle x^2 + 2x + 3\rangle$.
   (b) Let α be a root of $x^2 + 2x + 3$. Write down every element in $\mathbb{F}_{25}$ in terms of α and elements in $\mathbb{F}_5$.
   (c) Write down the two roots of $x^2 - 2$ in terms of α and elements in $\mathbb{F}_5$.
3. Prove that $\mathrm{Gal}(\mathbb{Q}[2^{1/3}, \mu_3]/\mathbb{Q}) = S_3$.
4. Let $f(x) = x^8 - 1 \in \mathbb{Q}[x]$.
   (a) Determine the splitting field K of $f(x)$.
   (b) Determine the degree $[K : \mathbb{Q}]$.
   (c) Determine all elements in $\mathrm{Aut}(K/\mathbb{Q})$ as ring homomorphisms.
   (d) Let $\alpha = \mu_8 + \mu_8^{-1}$ where $\mu_8 = e^{2\pi i/8}$. Determine $\sigma(\alpha)$ for every $\sigma \in \mathrm{Aut}(K//Q)$.
   (e) Determine the finite group $\mathrm{Gal}(K/\mathbb{Q})$.
5. (a) Determine the intermediate fields M_i with $\mathbb{Q}[2^{1/3}, \mu_3] \supset M \supset \mathbb{Q}$ and $\mathrm{Gal}(\mathbb{Q}[2^{1/3}, \mu_3]/M_i)$.
   (b) Determine the intermediate fields M_i between $K \supset M \supset \mathbb{Q}$ where K is the splitting field of $x^8 - 1$ and $\mathrm{Gal}(K/M_i)$.