Homework 4, Math 401

due on Feb 22, 2021

Before you start, please read the syllabus carefully.

1. Let A be an abelian group. Prove that

   (a) The set of group homomorphisms from $\mathbb{Z}$ to A forms an abelian group.

   (b) Call the group from the last step by B. There is an isomorphism between A and B.

2. Let G be a group, $N \triangleleft G$ be a normal subgroup and $H \subset G$ be a subgroup. Prove that the subgroup generated by N and H is

$$\{h \cdot n \mid h \in H, n \in N\}.$$

3. Let G be a group. Define $\sigma : G \times G \to G$ to be the conjugation, i.e., $\sigma(g, x) = gxg^{-1}$. Prove that σ is a group action of G on G.

4. Let G be a group, $H \subset G$ be its subgroup and X be the set of left cosets of H. Define $\sigma : G \times X \to X$ to be the left multiplication, i.e., $\sigma(g, xH) = gxH$. Prove that σ is a group action of G on X.

5. Let G be a group with order p-power where p is a prime number. Prove that the center of G is non-trivial.

6. Let G be a group with order p^2, then we know G must be abelian from the lecture. Define the exponent of G, denoted $\exp(G)$, to be smallest integer n such that $g^n = e$ for any $g \in G$.

   (a) Prove that $\exp(C_{p^2}) = p^2$ and $\exp(C_p \times C_p) = p$.

   (b) If $\exp(G) = p^2$, then $G \simeq C_{p^2}$.

   (c) If $\exp(G) = p$, then $G \simeq C_p \times C_p$.
   (Hint: First show that $G = \langle a, b\rangle$ for any $a \neq e$ and $b \notin \langle a\rangle$, then construct the isomorphism by specifying the images for a and b).

7. Let G be the group of order pq where $p \neq 1 \mod q$ and $q \neq 1 \mod p$ and both p and q are prime numbers.

   (a) Use Sylow's theorem to determine the number of Sylow-p subgroups and Sylow-q subgroups.

   (b) Prove that $G \simeq C_{pq}$.
   (Hint: Use the fact that the order of a group element must divide $|G|$, and count the number of elements with order p and q.)

8. Prove that all groups with order less or equal to 5 are abelian.

   (Hint: Write out all finite groups with order ≤ 5.)