Homework 5, Math 401

due on March 8, 2021

Before you start, please read the syllabus carefully.

1. For any ring R, prove that there exists a unique ring homomorphism from $\mathbb{Z}$ to R.
2. Find the number of group homomorphisms from A to B, and the number of ring homomorphisms from A to B (Just numbers.)
   (a) $A = \mathbb{Z}$, $B = C_6$
   (b) $A = C_6$, $B = \mathbb{Z}$
   (c) $A = C_6$, $B = C_3$
   (d) $A = C_3$, $B = C_6$
3. Let a be a rational number and define $f : \mathbb{Q}[x] \to \mathbb{Q}$ to be $f(g(x)) = g(a)$. Prove that f is a ring homomorphism.
4. Let R be a ring and S be a subset of R. We define the ideal generated by S to be the smallest ideal containing S.
   (a) Prove that $\langle 1 \rangle = R$.
   (b) Prove that $\langle S \rangle = \{\sum_{i \leq N} r_i s_i t_i \mid r_i, t_i \in R, s_i \in S\}$.
5. For the ring of integers, denote $I = \langle 5 \rangle$, and $J = \langle 3 \rangle$
   (a) Prove that the ideal generated by n is $\langle n \rangle = \{kn \mid k \in \mathbb{Z}\}$.
   (b) Compute $I + J$, $I \cdot J$ and $I \cap J$.
6. For the ring of integers:
   (a) Prove that any subgroup of $(\mathbb{Z}, +)$ is cyclic.
   (Hint: Use the fact that if $\gcd(a, b) = d$, then $\exists s, t \in \mathbb{Z}$ such that $as + bt = d$).
   (b) Prove that any subgroup of $(\mathbb{Z}, +)$ is also an ideal of $(\mathbb{Z}, +, \times)$.