Homework 6, Math 401

due on March 15, 2021

Before you start, please read the syllabus carefully.

1. Prove that $x^4 + 1$ is irreducible in $\mathbb{Q}[x]$.
2. Let $f(x) \in F[x]$ be a polynomial with coefficient in a field F. Prove that for any $a \in F$, $x - a | f(x)$ if and only if $f(a) = 0$.
3. Prove that $\{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\} \subset \mathbb{R}$ forms a field under usual addition and multiplication for real numbers.
4. Let $f(x) = x^4 - 3x^2 + 2$ and $g(x) = x^3 - 1$. Compute $\gcd(f(x), g(x))$, and find $s(x)$ and $t(x)$ in $\mathbb{Q}[x]$ such that $f(x)s(x) + g(x)t(x) = \gcd(f(x), g(x))$.
5. Consider the ring of polynomials $C_5[x]$ where the coefficient is C_5. Let $f(x) = x^4 + x^2 + 1$ and $g(x) = x^2 + 2$, find $q(x)$ and $r(x)$ in $C_5[x]$ such that

$$f(x) = g(x)q(x) + r(x), \deg(r(x)) < \deg(g(x)).$$