Homework 7, Math 401

due on March 22, 2021

Before you start, please read the syllabus carefully.

1. Let $f(x) = x^{p-1} + x^{p-2} + \cdots + x + 1 \in \mathbb{Q}[x]$. Prove that $f(x)$ is irreducible if and only if p is a prime number.
2. Let R be a commutative ring and I be an ideal. Prove that there is a bijection between ideals of R/I and ideals of R containing I.
3. Let $R = \mathbb{Z}/\langle p^k \rangle$ be a quotient ring, where p is a prime number and $k \geq 1$ is an integer. Find out all ideals of this ring R.
4. Let p be a prime. Prove that $(a+b)^p = a^p + b^p \in C_p$, and $(f(x)+g(x))^p = f(x)^p + g(x)^p \in C_p[x]$.

   (Hint: Prove that $p|\binom{p}{k}$ for $1 \leq k \leq p-1$.)
5. Let $f(x) = x^{p-1} + x^{p-2} + \cdots + x + 1 \in C_p[x]$. Find a decomposition of $f(x)$ into a product of irreducible polynomials.
6. Find the minimal polynomial of $\alpha = (1+\sqrt{2})^{1/3}$ in $\mathbb{Q}[x]$.
7. Find the minimal polynomial of $\sqrt{2}$ in $C_5[x]$ and $C_7[x]$.
8. Prove that $C_5[x]/\langle x^2 - 3 \rangle$ is a field, and then count the number of elements in $C_5[x]/\langle x^2 - 3 \rangle$.