Homework 8, Math 401

due on March 29, 2021

Before you start, please read the syllabus carefully.

1. Prove that the only ideal in a field F is 0 and F itself.
2. Prove that if $K \subset F$ is a field extension with $[F : K] = 1$, then $F = K$.
3. Denote $K = \mathbb{Q}[\sqrt{2} + \sqrt{5}] \subset \mathbb{C}$.
   (a) Prove that $K \subset F := \mathbb{Q}[\sqrt{2}][\sqrt{5}]$ (i.e. F is a field extension of $\mathbb{Q}[\sqrt{2}]$).
   (b) Determine $[F : \mathbb{Q}]$.
   (c) Denote $\alpha = \sqrt{2} + \sqrt{5}$. Find the minimal polynomial of α.
   (d) Prove that $1, \alpha, \alpha^2, \alpha^3$ are linearly independent over $\mathbb{Q}$.
   (e) Write α^4 as a linear combination of $\{1, \alpha, \alpha^2, \alpha^3\}$,i.e., find $a, b, c, d \in \mathbb{Q}$ such that
   $$\alpha^4 = a + b\alpha + c\alpha^2 + d\alpha^3.$$
   (f) Prove that $\mathbb{Q}[x]/\langle f(x)\rangle \simeq K$ where $f(x) = x^4 - (a + bx + cx^2 + d^3)$.
   (g) Prove that $\{1, \alpha, \alpha^2, \alpha^3\}$ is a basis for K as a vector space over $\mathbb{Q}$.
   (h) Prove that $K = F$.
4. Let $f(x) = x^2 - 2 \in \mathbb{F}_5[x]$, and denote K to be the splitting field of $f(x)$.
   (a) Determine $[K : \mathbb{F}_5]$.
   (b) Let $g(x) = x^2 - 3 \in K[x]$. Determine whether $g(x)$ is irreducible in $K[x]$, if yes, give a proof, if not, find its decomposition.
5. For an arbitrary field F, we define a formal operation on $f(x) \in F[x]$ called *derivative* as following
   $$f'(x) := \sum_n a_n \cdot n \cdot x^{n-1},$$
   if $f(x) = \sum_n a_n \cdot x^n$.
   (a) Prove that
   $$(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x),$$
   for $F[x]$ for arbitrary field F.
   (b) Prove that if $f(x)$ has a multiple root α, equivalently $(x - \alpha)^2 | f(x))$, then α is also a root of $f'(x)$.

(c) Does the converse from above holds? i.e., if $f'(\alpha) = 0$, does it imply that α is a multiple root $f(x)$? If yes, give a proof, if no, give a counter example or give a correct statement.

(d) (**Warning**: the following statement is only true when F has characteristic 0) Prove that $f(\alpha) = f'(\alpha) = \cdots = f^{(k)}(\alpha) = 0$ if and only if $(x-\alpha)^{k+1}|f(x)$ for $f(x)$ in polynomial ring $F[x]$ where F is an arbitrary field. (Does this remind you of Taylor expansion in calculus?)

6. (**Bonus Question**) Let L/F be field extension, and E is the subset of algebraic elements in L over F.

   (a) Prove that E is a field extension of F.

   (Hint: prove that if a and b are both algebraic, then $a+b$ and ab are also algebraic.)

   (b) Prove that if L is algebraic closed, then E is the algebraic closure of F.

   (Hint: prove E is algebraic, and then prove it is algebraically closed.)