Homework 9, Math 401

due on April 5, 2021

Before you start, please read the syllabus carefully.

1. Find all ring homomorphisms from $\mathbb{F}_p$ to $\mathbb{F}_p$.

2. (a) Prove that $\pi : x \to x^p$ is a ring isomorphism from $\mathbb{F}_q$ to $\mathbb{F}_q$ where $q = p^k$ and p is a prime number.

   (b) Prove that for any integer $1 \leq r \leq k$, π^r (r-th composition with itself): $\mathbb{F}_q \to \mathbb{F}_q$ is also a ring isomorphism.

   (c) Prove that $\mathrm{Aut}(\mathbb{F}_9/\mathbb{F}_3) = C_2$.

3. Find a decomposition of $x^q - x \in \mathbb{F}_p[x]$ when $q = p^2$ for a prime number p.

4. Let $f(x) \in F[x]$ be irreducible with degree n.

   (a) If $f(x)$ and $f'(x)$ are relatively prime, prove that $f(x)$ has no repeated roots.

   (b) If $\mathrm{char}(F) = 0$, prove that $f'(x)$ has degree $n - 1$ and $f(x)$ and $f'(x)$ are relatively prime.

   (c) If $F = \mathbb{F}_p$, prove that $f(x)$ has no repeated roots.
   (Hint:Consider the splitting field of $f(x)$)

5. Prove that $\mathbb{Q}[\mu_3 + 2^{1/3}] = \mathbb{Q}[\mu_3, 2^{1/3}]$.