Homework 10, Math 3000

due on April 12, 2022

Before you start, please read the syllabus carefully.

1. Let $V := \{a_0 + a_1 t + a_2 t^2 + a_3 t^3 \mid a_i \in \mathbb{R}\}$ be the vector space of polynomials with degree smaller or equal to 3. We define a new dot product for V to be $f * g := \int_0^1 (f \cdot g) dt$. For example $t * t^2 = \int_0^1 t^3 dt = 1/4$.
   (a) Compute $t^i * t^j$.
   (b) Apply Gram-Schmidt to find an orthonormal basis for V with respect to $*$.
   Hint: replace $u \cdot v$ by $f * g$ in Gram-Schmidt.
2. Let $V = \mathbb{R}^3$, and $W_1 := \mathrm{span}((1,0,1),(0,1,0))$ and $W_2 := \mathrm{span}((0,0,1))$.
   (a) Show that $W_1 \cap W_2 = 0$.
   (b) Show that $W_1 \oplus W_2 = V$.
3. Let U be the subspace of $\mathbb{R}^3$ be spanned by $(1,1,1)$.
   (a) Determine elements of the quotient space $Q = \mathbb{R}^3/U$.
   (b) Find all (a,b,c) such that $(a,b,c) + U$ contains the origin.
   (c) Determine the dimension of Q.
4. Let V be the vector space of polynomials with degree smaller or equal to n.
   (a) Let W be the set of polynomials where $a_i = 0$ for $i \leq 2$. Prove that W is a subspace.
   (b) Simplify $(1+t)^{100} + W$ in the quotient space V/W.
5. Let $V = M_{n\times n}(\mathbb{R})$. Let $T : V \to V$ be the map $T(X) = AX - XA$ where A is a fixed matrix.
   (a) Show that $W := \{X \mid \mathrm{tr}(X) = 0\}$ is an invariant space for T.
   (b) Find a basis of W and determine the dimension of W.
   (c) For $n = 2$ and
   $$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},$$
   write down the matrix for T with respect to the basis you have chosen for the last question.
6. Let $V = \mathbb{R}^2$ and $W_1 := \mathrm{span}(\mathbf{u})$ where $\mathbf{u} = (1,2)$.
   (a) Let $T : V \to V$ be the map that $T(\mathbf{v}) = \mathrm{Proj}_{\mathbf{u}}(\mathbf{v})$. Determine $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$.

(b) Prove that $\mathbb{R}^2 = \mathrm{Ker}(T) \oplus \mathrm{Im}(T)$.

7. Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find all generalized eigenvectors of A.