Homework 11, Math 3000

due on April 26, 2022

Before you start, please read the syllabus carefully.

1. For each linear map $T : V \to V$, find a basis of V consisting of generalized eigenvectors for T.

   (a) Let V be $\mathbb{R}^3$, and T is represented by

   $$A = \begin{pmatrix} -1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

   under standard basis of $\mathbb{R}^3$.

   (b) Let V be $\mathbb{R}^4$, and T is represented by

   $$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ -1 & 3 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

   under standard basis of $\mathbb{R}^4$.

2. (a) Let V be $\mathbb{R}^3$ and $T : V \to V$ a linear map. Suppose for some $\mathbf{v} \in V$, we have $\{(T-\lambda I)^2\mathbf{v}, (T-\lambda I)\mathbf{v}, \mathbf{v}\}$ forms a basis of V, write down the Jordan canonical form of T.

   (b) If $A \sim B$ (recall A is similar to B if $B = C^{-1}AC$ for some C), then show that $\dim(\mathrm{Ker}(A-\lambda I)^k) = \dim(\mathrm{Ker}(B-\lambda I)^k)$ for all $k \geq 0$.

   (c) Prove that the following two matrices are not similar to each other:

   $$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

   (Hint: use (b).)

   (d) Find out all similar classes of $M_{3\times 3}(\mathbb{C})$ with characteristic polynomial $(\lambda-1)^3$. (That is, give a list of matrices where each two of them are not similar to each other.)

3. Determine the Jordan canonical form of the following matrix and write down the minimal polynomial for A.

(a)

$$A = \begin{pmatrix} -4 & 9 \\ -4 & 8 \end{pmatrix}$$

(b)

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{pmatrix}$$

4. (a) If a matrix $A \in M_{3\times 3}(\mathbb{C})$ has characteristic polynomial $(\lambda-1)^3$ and minimal polynomial $(\lambda-1)^2$, can you write down the Jordan canonical form?

   (b) If $A \in M_{4\times 4}(\mathbb{C})$ has only one eigenvalue $\lambda = 1$, and $\mathrm{Ker}(A-I)$ has dimension 3, can you write down the Jordan canonical form of A?

   (c) Find out all similar classes of $M_{4\times 4}(\mathbb{C})$ with characteristic polynomial $(\lambda-1)^4$ and minimal polynomial $(\lambda-1)^2$.

5. (a) If A has characteristic polynomial $(\lambda-1)^5$, and $\mathrm{Ker}(A-I)$ has dimension 1, what are the possible dimensions for $\mathrm{Ker}((A-I)^2)$?

   (b) If $\mathrm{Ker}(A-I)$ has dimension 2, then what are the possible dimensions for $\mathrm{Ker}(A-I)^2$?

6. Let A be

$$\begin{pmatrix} -3/2 & 1 \\ -4 & 5/2 \end{pmatrix}.$$

   (a) Find the Jordan canonical form B of A.

   (b) Compute A^{100}. (Hint: compute B^{100})

   (c) What is $\lim_{k\to\infty} A^k\mathbf{x}$ where $\mathbf{x} = (1,1)^T$?