Homework 9, Math 3000

due on April 5, 2022

Before you start, please read the syllabus carefully.

1. Apply Gram-Schmidt algorithm to find an orthonormal set of vectors from the following set of vectors.

   (a) $\{(0,1,0),(1,2,1),(2,1,2)\}$

   (b) $\{(0,1,0,1),(1,0,1,0),(1,1,1,1)\}$

2. Find an orthogonal matrix C such that $C^{-1}AC$ is diagonal for the following matrices:

   (a)
   $$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

   (b)
   $$A = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

   (c)
   $$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

3. Define *trace* of a matrix to be $\sum_i A_{ii}$, denote it to be $\mathrm{Tr}(A)$.

   (a) Prove that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

   (b) Show that if A is similar to B (i.e. $C^{-1}AC = B$ for some C), then $\mathrm{Tr}(A) = \mathrm{Tr}(B)$.

4. Prove that if A is similar to B, then $\sum_n a_n A^n = 0$ if and only if $\sum_n a_n B^n = 0$.

5. Let V be a vector space. Let $S_i := \{v_{i1}, \cdots, v_{in_i}\}$ be set of vectors in V, for $i = 1, \cdots, m$. Suppose for all i, S_i is linearly independent, and $\{w_1, \cdots, w_m\}$ is linearly independent for any $w_i \in \mathrm{span}(S_i)$. Prove that $S_1 \cup \cdots \cup S_m$ is linearly independent.

6. Consider $M \in M_{2\times 2}(\mathbb{R})$ with
   $$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

   (a) Write down the characteristic polynomial f of M.

   (b) If $4bc = -(a-d)^2$, then prove that M has two different eigenvalues.

(c) If the characteristic polynomial for M_1 and M_2 is the same, and both satisfy $4bc = -(a-d)^2$, then show that M_1 is similar to M_2.

(d) Construct one example where M_1 and M_2 have the same characteristic polynomial, but M_1 is not similar to M_2.

(e) Suppose the polynomial for M is $f(\lambda) = \sum_n a_n \lambda^n$. Prove that $\sum_n a_n M^n = 0$.

7. Determine the eigenvalue and eigenvectors for the following matrix (possibly use complex numbers $\mathbb{C}$)

(a)

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

(b)

$$A = \begin{pmatrix} -1 & -2 \\ 1 & 0 \end{pmatrix}$$